Turing Machines

Part Two

Outline for Today

- ***The Church-Turing Thesis***
  - Just how powerful are TMs?
- ***What Does it Mean to Solve a Problem?***
  - Rethinking what “solving” a problem means, and two possible answers to that question.

Recap from Last Time

Turing Machines

- A ***Turing machine*** is a program that controls a tape head as it moves around an infinite tape.
- There are six commands:
  - **`Move`** *direction*
  - **`Write`** *symbol*
  - **`Goto`** *label*
  - **`Return`** *boolean*
  - **`If`** *symbol command*
  - **`If Not`** *symbol command*
- Despite their limited vocabulary, TMs are surprisingly powerful.

A Sample Turing Machine

- Here's a sample TM.
- It receives inputs over the alphabet Σ = {**a**, **b**}.
- What strings does this TM accept?
- Can you write a regex that matches precisely the strings this TM accepts?

```
Start:
    If Not 'a' Return False

Loop:
    Move Right
    If Not Blank Goto Loop
    Move Left
    Move Left
    If Not 'b' Return False
    Return True
```

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What Can We Do With a TM?

- Last time, we saw TMs that
  - check if a string has the form **a**^{n}**b**^{n},
  - check if a string has the same number of **a**'s and **b**'s and
  - sort a string of **a**'s and **b**'s.
- Here's a list of some other things TMs can do; we'll give you these TMs with the starter files for PS8 this week.
  - Check if a number is a Fibonacci number.
  - Convert the number n into a string of n **a**'s.
  - Check if a string is a *tautonym* (the same string repeated twice).
  - So much more!
- This hints at the idea that TMs might be more powerful than they look.

New Stuff!

Main Questions for Today:

Just how powerful are Turing machines?

What problems can you solve with a computer?

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Just how powerful are Turing machines?

What problems can you solve with a computer?

Real and “Ideal” Computers

- A real computer has memory limitations: you have a finite amount of RAM, a finite amount of disk space, etc.
- However, as computers get more and more powerful, the amount of memory available keeps increasing.
- An ***idealized computer*** is like a regular computer, but with unlimited RAM and disk space. It functions just like a regular computer, but never runs out of memory.

Theorem: Turing machines are equal in power to idealized computers. That is, any computation that can be done on a TM can be done on an idealized computer and vice-versa.

Key Idea: Two models of computation are equally powerful if they can simulate each other.

Simulating a TM

- The individual commands in a TM are simple and perform only basic operations:

```
Move   Write   Goto   Return   If
```

- The memory for a TM can be thought of as a string with some number keeping track of the current index.
- To simulate a TM, we need to
  - see which line of the program we're on,
  - determine what command it is, and
  - simulate that single command.
- ***Claim:*** An idealized computer can simulate a TM.
  - The "core" logic for the TM simulator is under fifty lines of code, including comments.

Simulating a TM

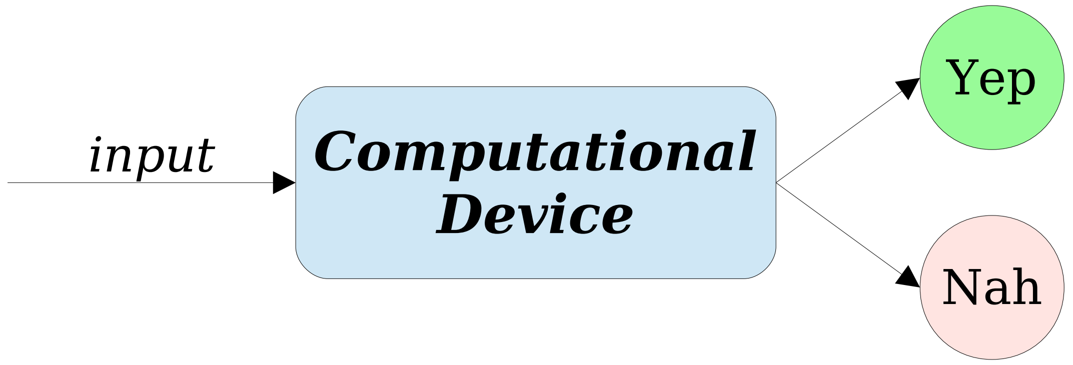
- Because a computer can simulate each individual TM instruction, an idealized computer can do anything a TM can do.
- ***Key Idea:*** Even the most complicated TM is made out of individual instructions, and if we can simulate those instructions, we can simulate an arbitrarily complicated TM.

Simulating a Computer

- Programming languages provide a set of simple constructs.
  - Think things like variables, arrays, loops, functions, classes, etc.
- You, the programmer, then combine these basic constructs together to assemble larger programs.
- ***Key Idea:*** A TM is powerful enough to simulate each of these individual pieces. It’s therefore powerful enough to simulate anything a real computer can do.

A Leap of Faith

- ***Claim:*** A TM is powerful enough to simulate any computer program that gets an input, processes that input, then returns some result.



- The resulting TM might be colossal, or really slow, or both, but it would still faithfully simulate the computer.
- We're going to take this as an article of faith in CS103. If you curious for more details, come talk to me after class.

Can a TM Work With…

"cat pictures?"

Sure! A picture is just a 2D array of colors, and a color can be represented as a series of numbers.

Can a TM Work With…

~~“cat pictures?”~~

“cat videos?”

If you think about it, a video is just a series of pictures!

Can a TM Work With…

"music?"

Sure! Music is encoded as a compressed waveform. That's just a list of numbers.

"ChatGPT?"

Sure! That's just applying a bunch of matrices and nonlinear functions to some input.

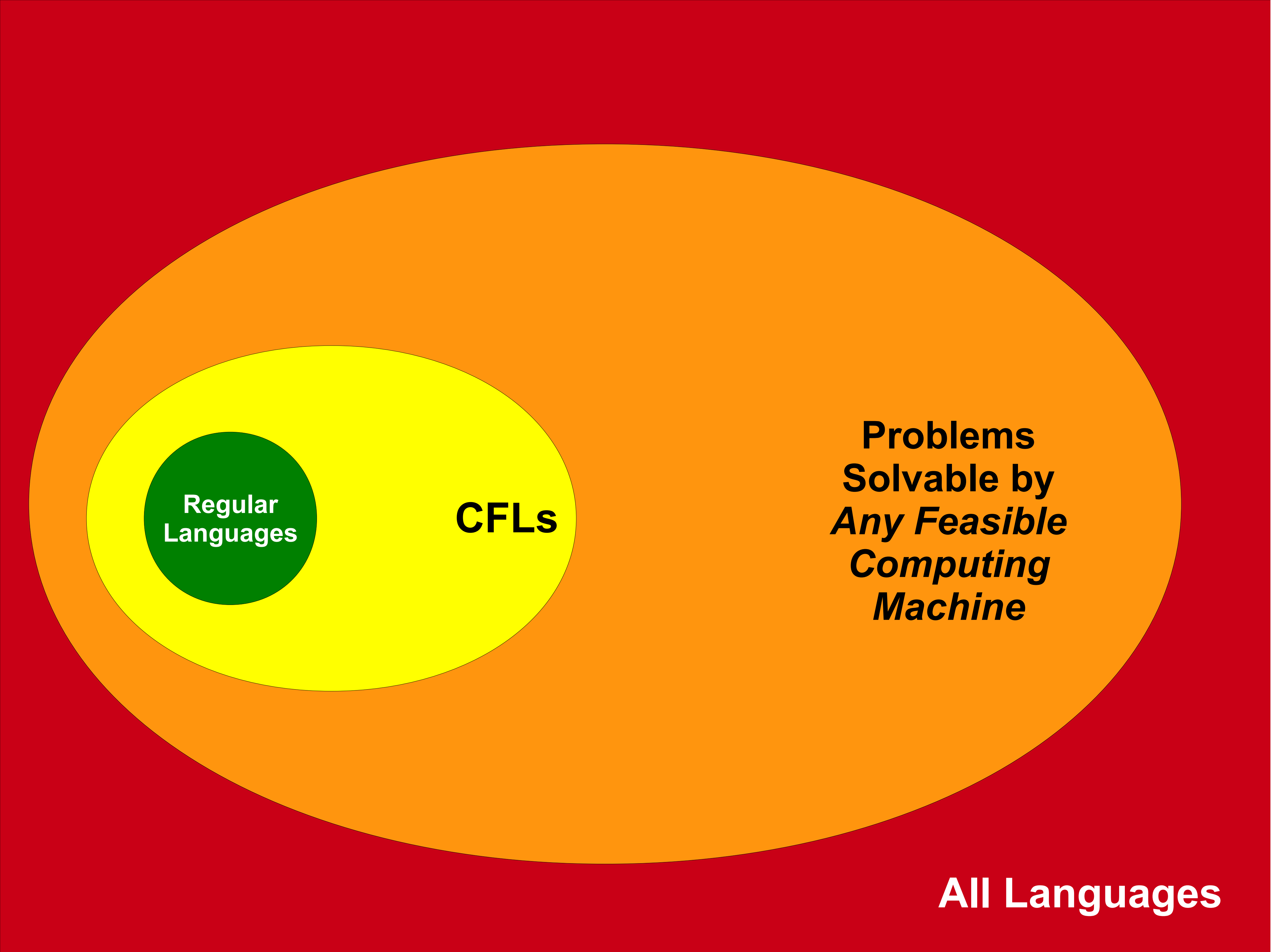
Just how powerful ***are*** Turing machines?

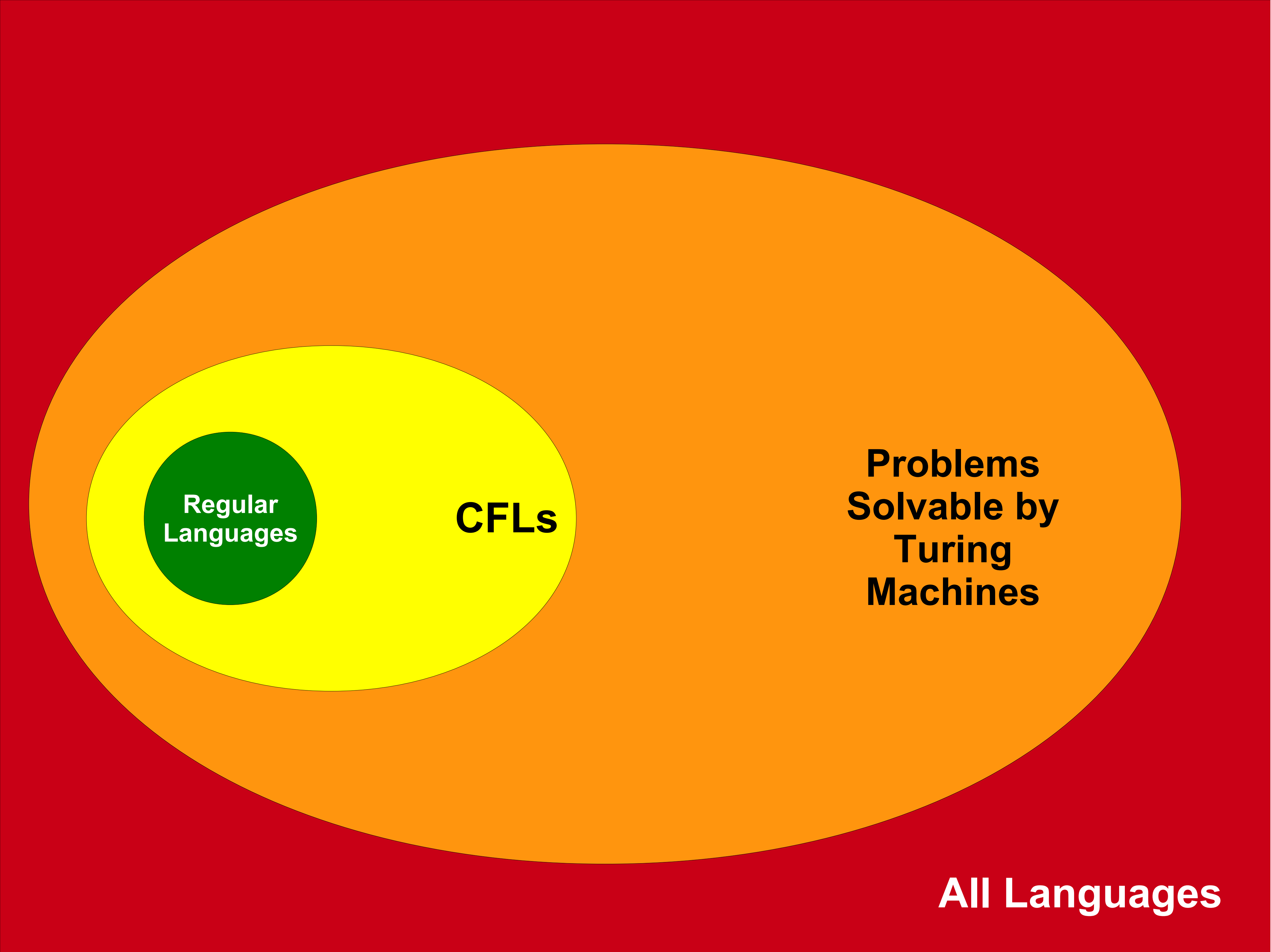
The ***Church-Turing Thesis*** claims that

every feasible method of computation is either equivalent to or weaker than a Turing machine.

“This is not a theorem – it is a falsifiable scientific hypothesis. And it has been thoroughly tested!”

- Ryan Williams


Regular Languages
CFLs
Problems Solvable by *Any Feasible Computing Machine*
All Languages


Regular Languages
CFLs
Problems Solvable by Turing Machines
All Languages

TMs and Computation

- Because Turing machines have the same computational powers as regular computers, we can (essentially) reason about Turing machines by reasoning about actual computer programs.
- Going forward, we're going to switch back and forth between TMs and computer programs based on whatever is most appropriate.
- In fact, our eventual proofs about the existence of impossible problems will involve a good amount of pseudocode. Stay tuned for details!

Time-Out for Announcements!

Second Midterm Logistics

- Our second midterm exam is ***tomorrow, May 20th*** from ***6-9 PM***.
- Topic coverage is primarily lectures 06 – 15 (functions through DFAs & NFAs) and PS3 – PS5 and the first two questions of PS6.
  - Because the material is cumulative, topics from PS1 – PS2 and Lectures 00 – 05 are also fair game.
- The exam is closed-book and closed-computer. You can bring one double-sided 8.5” × 11” sheet of notes with you.

You should have received email about your room and seat for the exam.

Back to CS103!

Decidability and Recognizability

What problems can we solve with a computer?

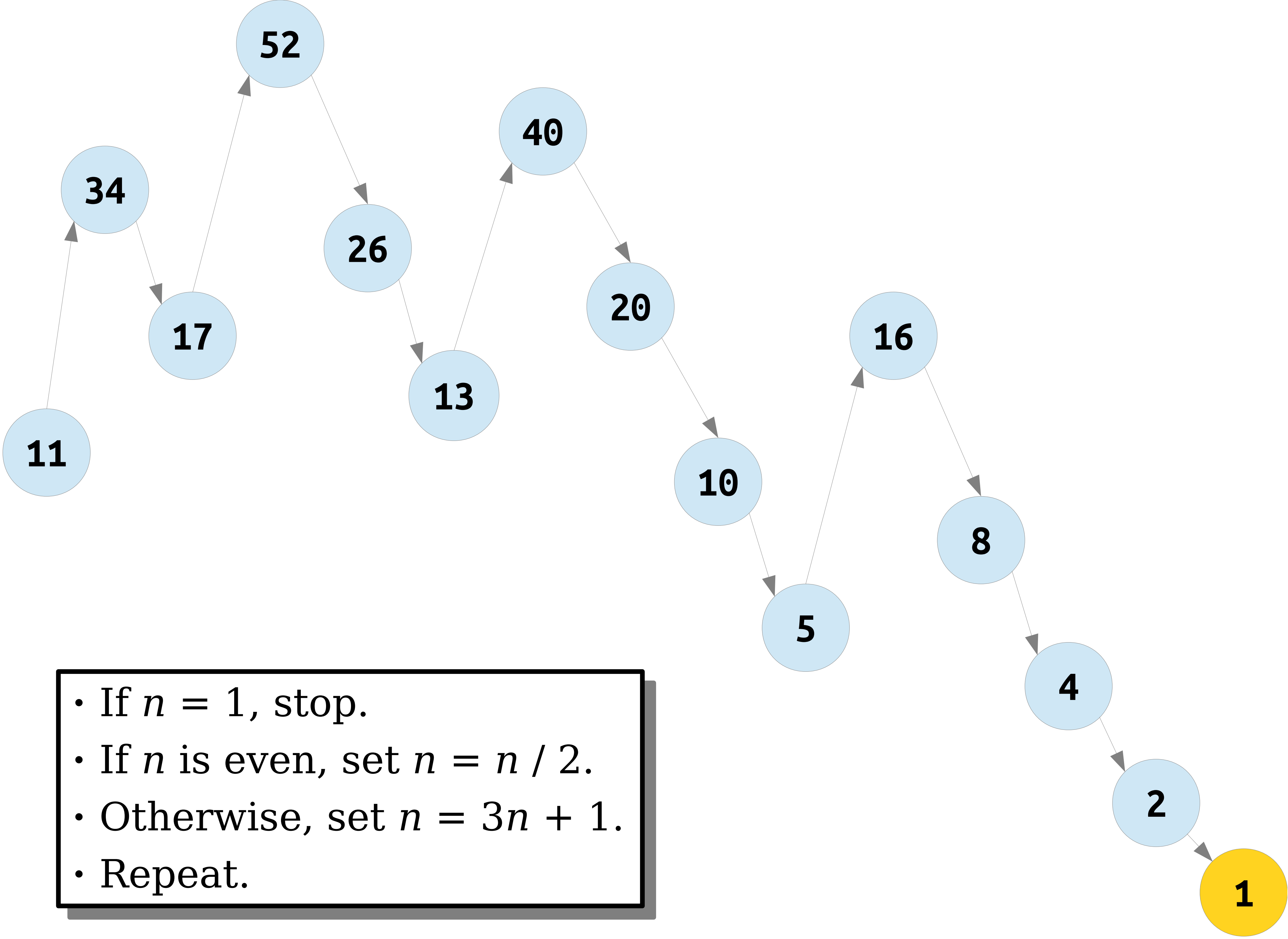
What kind of computer?

What problems can we solve with a computer?

What does it mean to "solve" a problem?

The Hailstone Sequence

- Consider the following procedure, starting with some $n \in \mathbb{N}$, where $n > 0$:
  - If $n = 1$, you are done.
  - If n is even, set $n = n / 2$.
  - Otherwise, set $n = 3n + 1$.
  - Repeat.
- ***Question:*** Given a natural number $n > 0$, does this process terminate?



- If $n = 1$, stop.
- If n is even, set $n = n / 2$.
- Otherwise, set $n = 3n + 1$.
- Repeat.

The Hailstone Sequence

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  - If n is even, set $n = n / 2$.
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  - Repeat.
- Does the Hailstone Sequence terminate for…
  - $n = 5$?
  - $n = 20$?
  - $n = 7$?
  - $n = 27$?

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The Hailstone Turing Machine

- Let $\Sigma = \{$`a`$\}$ and consider the language

  $L = \{$ `a`$^n \mid n > 0$ and the hailstone sequence terminates for n $\}$.

- We can build a TM for L as follows:

```
If the input is ε, reject.
While the string is not a:
    If the input has even length, halve the length of the string.
    If the input has odd length, triple the length of the string and append a a.
Accept.
```

Does this Turing machine accept all nonempty strings?

The Collatz Conjecture

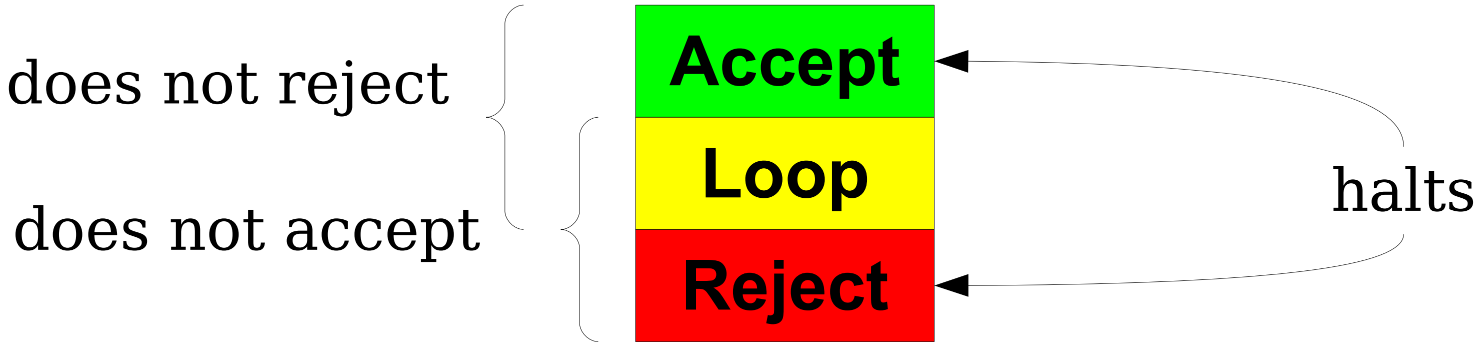
- It is *unknown* whether this process will terminate for all natural numbers.
  - In other words, no one knows whether this TM always terminates!
- The conjecture (unproven claim) that the hailstone sequence always terminates is called the ***Collatz Conjecture***.
- This problem has eluded a solution for a long time. The influential mathematician Paul Erdős is reported to have said “mathematics may not be ready for such problems.”

An Important Observation

- Unlike finite automata, which automatically halt after all the input is read, TMs keep running until they explicitly return true or return false.
- As a result, it’s possible for a TM to run forever without accepting or rejecting.
- This leads to several important questions:
  - How do we formally define what it means to build a TM for a language?
  - What implications does this have about problem-solving?

Very Important Terminology

- Let M be a Turing machine.
- M ***accepts*** a string w if it returns true on w.
- M ***rejects*** a string w if it returns false on w.
- M ***loops infinitely*** (or just ***loops***) on a string w if when run on w it neither returns true nor returns false.
- M ***does not accept w*** if it either rejects w or loops on w.
- M ***does not reject w*** w if it either accepts w or loops on w.
- M ***halts on w*** if it accepts w or rejects w.



Recognizers and Recognizability

- A TM M is called a ***recognizer*** for a language L over Σ if the following statement is true:

$$\forall w \in \Sigma^*. \ (w \in L \ \leftrightarrow \ M \text{ accepts } w)$$

- A language L is called ***recognizable*** if there is a recognizer for it.
- If you are absolutely certain that $w \in L$, then running a recognizer for L on w will (eventually) confirm this.
  - Eventually, M will accept w.
- If you don't know whether $w \in L$, running M on w may never tell you anything.
  - M might loop on w – but you can't differentiate between "it'll accept if you wait longer" and "it will never come back with an answer."
- Does this feel like "solving a problem" to you?

Recognizers and Recognizability

- The hailstone TM M we saw earlier is a recognizer for the language

$$L = \{\ \mathtt{a}^n \mid n > 0 \text{ and the hailstone sequence terminates for } n\ \}.$$

- If the sequence does terminate starting at n, then M accepts $\mathtt{a}^n$.
- If the sequence doesn't terminate, then M loops forever on $\mathtt{a}^n$ and never gives an answer.
- If you somehow knew the hailstone sequence terminated for n, this machine would (eventually) confirm this. If you didn't know, this machine might not tell you anything.

Recognizers and Recognizability

- Earlier this quarter you explored sums of five cubes. Now, let's talk about sums of three cubes.
- Are there integers x, y, and z where…
  - $x^3 + y^3 + z^3 = 10$?
  - $x^3 + y^3 + z^3 = 11$?
  - $x^3 + y^3 + z^3 = 12$?
  - $x^3 + y^3 + z^3 = 13$?

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Recognizers and Recognizability

- Earlier this quarter you explored sums of five cubes. Now, let's talk about sums of three cubes.
- Are there integers *x*, *y*, and *z* where…
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Recognizers and Recognizability

- Surprising fact: until 2019, no one knew whether there were integers *x*, *y*, and *z* where

$$x^3 + y^3 + z^3 = 33.$$

- A heavily optimized computer search found this answer:

$$x = 8{,}866{,}128{,}975{,}287{,}528$$
$$y = -8{,}778{,}405{,}442{,}862{,}239$$
$$z = -2{,}736{,}111{,}468{,}807{,}040$$

- As of early 2025, no one knows whether there are integers *x*, *y*, and *z* where

$$x^3 + y^3 + z^3 = 114.$$

Recognizers and Recognizability

- Consider the language

$$L = \{\, \mathtt{a}^n \mid \exists x \in \mathbb{Z}.\ \exists y \in \mathbb{Z}.\ \exists z \in \mathbb{Z}.\ x^3 + y^3 + z^3 = n \,\}$$

- Here's pseudocode for a recognizer to see whether such a triple exists:

```
for max = 0, 1, 2, …
  for x from -max to +max:
    for y from -max to +max:
      for z from -max to +max:
        if $x^3 + y^3 + z^3 = n$: return true
```

- If you somehow knew there was a triple *x*, *y*, and *z* where $x^3 + y^3 + z^3 = n$, running this program will (eventually) convince you of this.
- If you weren't sure whether a triple exists, this recognizer might not be useful to you.

Recognizers and Recognizability

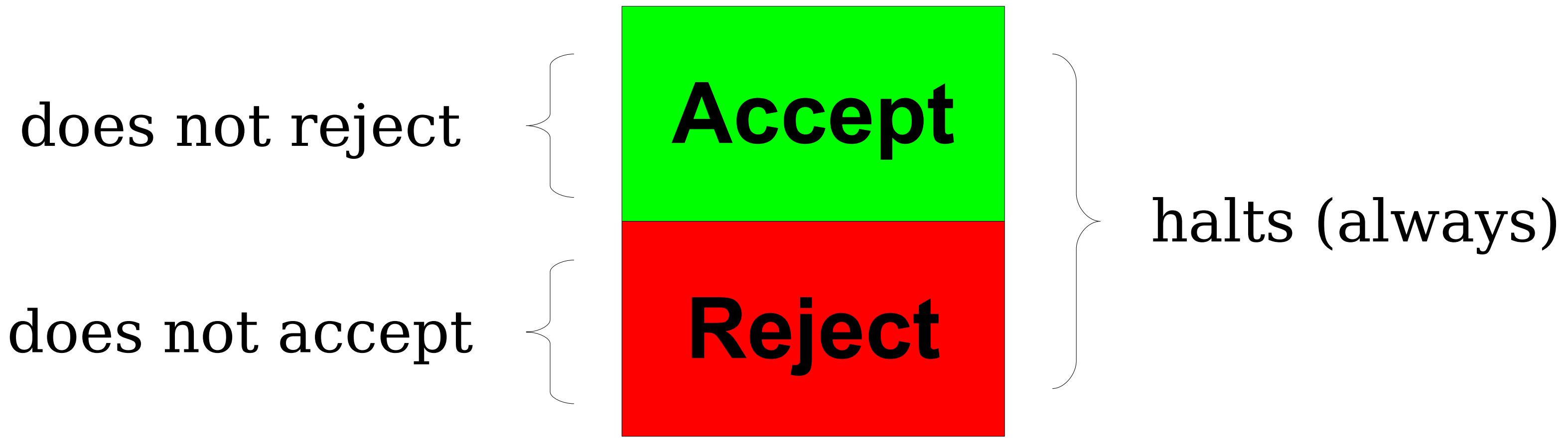
- The class **RE** consists of all recognizable languages.
- Formally speaking:

  RE = { L | L is a language and there's a recognizer for L }

- You can think of **RE** as "all problems with yes/no answers where "yes" answers can be confirmed by a computer."
  - Given a recognizable language L and a string $w \in L$, running a recognizer for L on w will eventually confirm $w \in L$.
  - The recognizer will never have a "false positive" of saying that a string is in L when it isn't.
- This is a "weak" notion of solving a problem.
- Is there a "stronger" one?

Deciders and Decidability

- Some, but not all, TMs have the following property: the TM halts on all inputs.
- If you are given a TM M that always halts, then for the TM M, the statement "M does not accept w" means "M rejects w."



Deciders and Decidability

- A TM M is called a ***decider*** for a language L over Σ if the following statements are true:

$$\forall w \in \Sigma^*.\ M \text{ halts on } w.$$

$$\forall w \in \Sigma^*.\ (w \in L \quad \leftrightarrow \quad M \text{ accepts } w)$$

- A language L is called ***decidable*** if there is a decider for it.
- A decider M for a language L accepts all strings in L and rejects all strings not in L.
- A decider M for a language L is a recognizer for L that halts on all inputs.
- Intuitively, if you don't know whether $w \in L$, running M on w will "create new knowledge" by telling you the answer.
- This is a "strong" notion of "solving a problem."

Deciders and Decidability

- The hailstone TM M we saw earlier is a ***recognizer*** for the language

$$L = \{ \mathbf{a}^n \mid n > 0 \text{ and the hailstone sequence terminates for } n \}.$$

- If the hailstone sequence terminates for n, then M accepts $\mathbf{a}^n$. If it doesn't, then M does not accept $\mathbf{a}^n$.
- We honestly don't know if M is a decider for this language.
  - If the hailstone sequence always terminates, then M always halts and is a decider for L.
  - If the hailstone sequence doesn't always terminate, then M will loop on some inputs and isn't a decider for L.

Deciders and Decidability

- While no one knows whether there are integers *x*, *y*, and *z* where

  $$x^3 + y^3 + z^3 = 114,$$

  it is very easy to figure out whether there are integers *x*, *y*, and *z* where

  $$x^2 + y^2 + z^2 = 114.$$

- Take a minute to discuss – why is this?

Deciders and Decidability

- Consider the language

  $L = \{ \texttt{a}^n \mid \exists x \in \mathbb{Z}.\ \exists y \in \mathbb{Z}.\ \exists z \in \mathbb{Z}.\ x^2 + y^2 + z^2 = n \}.$

- Here's pseudocode for a decider to see whether such a triple exists:

```
for x from 0 to n:
  for y from 0 to n:
    for z from 0 to n:
      if $x^2 + y^2 + z^2 = n$: return true
return false
```

- After trying all possible options, this program will either find a triple that works or report that none exists.

Deciders and Decidability

- The class **R** consists of all decidable languages.
- Formally speaking:

  $\mathbf{R} = \{ L \mid L \text{ is a language and there's a decider for } L \}$

- You can think of **R** as "all problems with yes/no answers that can be fully solved by computers."
  - Given a decidable language, run a decider for *L* and see what happens.
  - Think of this as "knowledge creation" – if you don't know whether a string is in *L*, running the decider will, given enough time, tell you.
- The class **R** contains all the regular languages, all the context-free languages, most of CS161, etc.
- This is a "strong" notion of solving a problem.

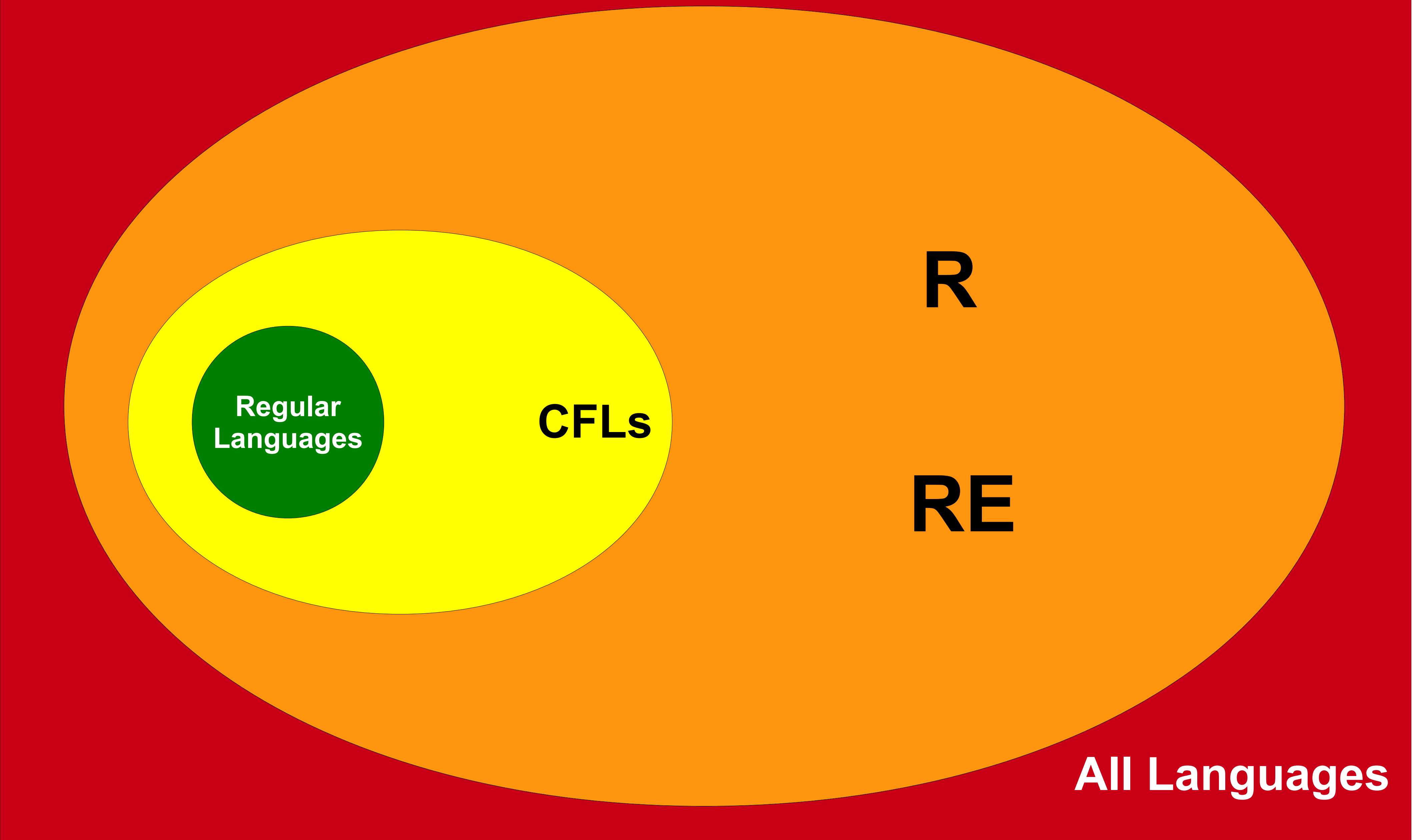
R and **RE** Languages

- Every decider for L is also a recognizer for L.
- This means that $\mathbf{R} \subseteq \mathbf{RE}$.
- Hugely important theoretical question:

R ʻP **RE**

- That is, if you can just confirm “yes” answers to a problem, can you necessarily *solve* that problem?

Which Picture is Correct?



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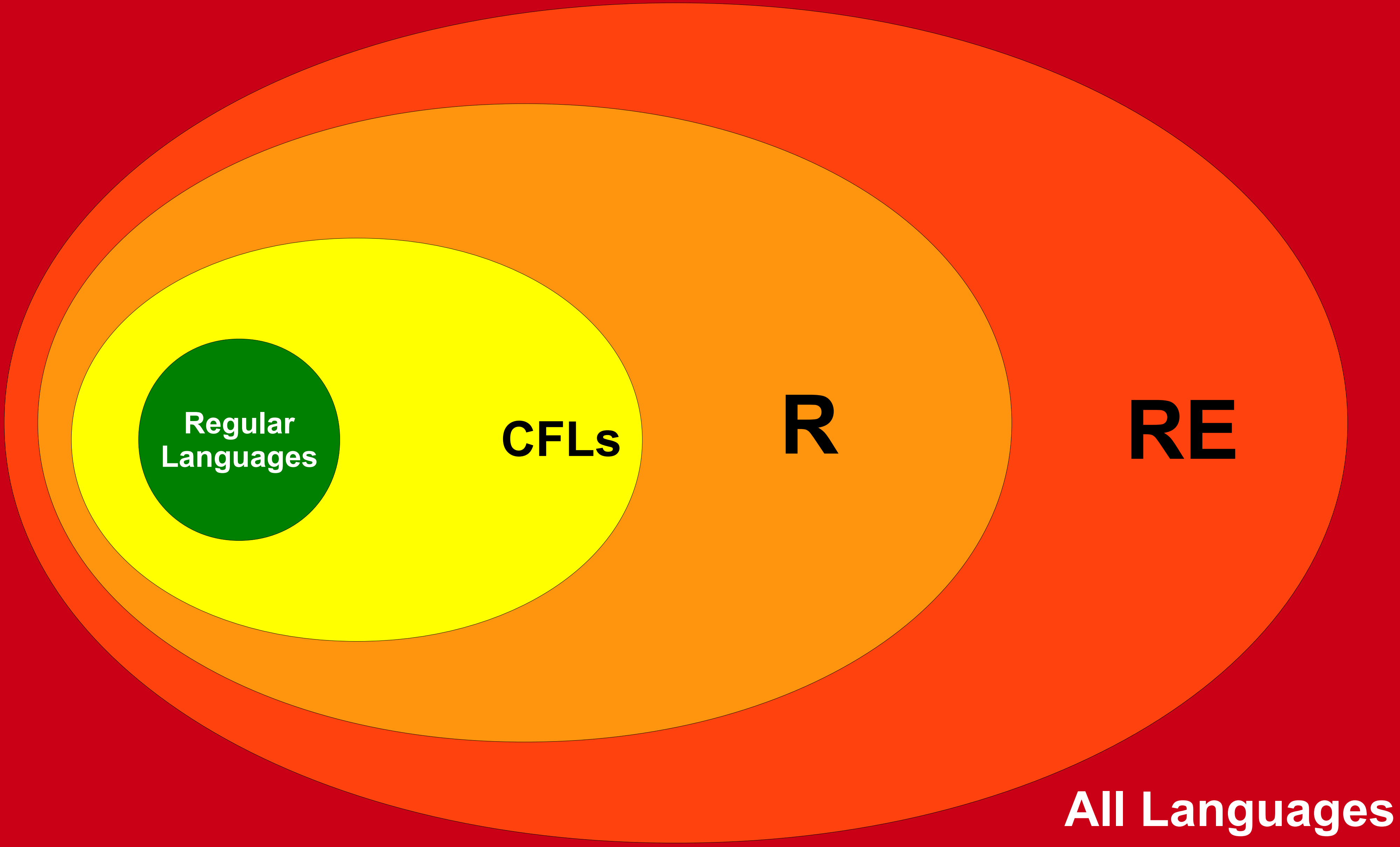

Unanswered Questions

- Why exactly is **RE** an interesting class of problems?
- What does the **R** $\overset{?}{=}$ **RE** question mean?
- Is **R** = **RE**?
- What lies beyond **R** and **RE**?
- We'll see the answers to each of these in due time.

Next Time

- ***Emergent Properties***
  - Larger phenomena made of smaller parts.
- ***Universal Machines***
  - A single, “most powerful” computer.
- ***Self-Reference***
  - Programs that ask questions about themselves.